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Phil. Trans. R. Soc. Lond. A 1967 **261**, 455-460
doi: 10.1098/rsta.1967.0013

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IX. Flow of current in a cross connected m.h.d. generator with four electrodes

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The current flow patterns in an m.h.d. generator with four electrodes are derived. There are two electrodes each side of the duct; the upstream one on the cathode side is connected to the downstream one on the anode side and a load across the other pair of electrodes produces a rudimentary cross-connected generator. The fluid properties, flow conditions and magnetic field are supposed uniform across the duct and a two-dimensional analysis is made. A Schwarz–Christoffel transformation is employed to find potential and current distributions. Three simultaneous linear equations must be solved to give explicit equations for currents and potentials. The method is extendable to the case of more electrodes.

1. Formulation of the problem

The generator considered has four electrodes, two on each opposing wall. The duct is of fixed width d, the electrodes are all of length λ and the spacing of each pair is ρ. Figure 1 shows the arrangement with electrodes numbered to distinguish them. Electrodes 1 and 3 are connected together and a load, of resistance R, is connected between electrodes 2 and 4.

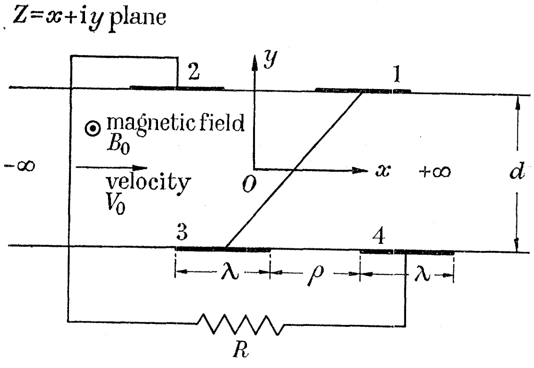

Figure 1.

The electrically conducting medium moves in the positive x direction with velocity V_0. The magnetic field B_0 acts in the positive z direction and so the electric field induced by the motion is perpendicular to both these directions in the negative y direction. The magnetic Reynolds number is low, so that the magnetic field may be taken as B_0 at every point; it is supposed that the conductivity σ, V_0, B_0 and the electron mobility μ are constant throughout the generator (as is assumed, for example, by Hurwitz, Kilb & Sutton 1961). Current flow need only be considered in two dimensions. We can express Ohm law, taking into account the Hall effect, as

$$\left.\begin{aligned} j_x &= \frac{\sigma}{1+\beta^2}(E'_x-\beta E'_y), \\ j_y &= \frac{\sigma}{1+\beta^2}(E'_y+\beta E'_x), \end{aligned}\right\} \tag{1}$$

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where β, the Hall parameter is given by

$$\beta = \mu B = \omega\tau,$$

$\mathbf{j}$ is the current and $\mathbf{E}'$ is the electric field as seen in the frame of reference of the moving medium.

$$\mathbf{E}' = -\nabla\phi + \mathbf{V}_0 \times \mathbf{B}_0, \tag{2}$$

where ϕ is the electrostatic potential. It is possible to define $F' = \phi' + \mathrm{i}\psi'$ where F' is an analytic function of $Z = x + \mathrm{i}y$ such that

$$E'_x = -\frac{\partial\phi'}{\partial x} = -\frac{\partial\psi'}{\partial y}, \tag{3a}$$

$$E'_y = -\frac{\partial\phi'}{\partial y} = \frac{\partial\psi'}{\partial x}. \tag{3b}$$

Hence, integrating relations (2) and (3) and matching arbitrary constants one may write

$$\phi' = \phi + V_0 B_0 y. \tag{4}$$

Then the current is written

$$\mathbf{j} = j_x - \mathrm{i}j_y. \tag{5}$$

The Ohm law can be represented by the complex equation:

$$\left.\begin{aligned} \mathbf{j} &= \frac{\sigma}{1+\beta^2}(1-\mathrm{i}\beta)E' \\ &= -\frac{\sigma}{(1+\beta^2)^{\frac{1}{2}}}\,\mathrm{e}^{-\mathrm{i}\theta}\,\frac{\mathrm{d}F'}{\mathrm{d}Z}, \end{aligned}\right\} \tag{6}$$

where

$$\mathbf{E}' = E'_x - \mathrm{i}E'_y$$

and where $\theta = \arctan\beta$. In terms of $F' = \phi' + \mathrm{i}\psi'$ the components of $\mathbf{j}$ may be written

$$j_x = \frac{\sigma}{1+\beta^2}\left(-\frac{\partial\psi'}{\partial y} + \beta\frac{\partial\phi'}{\partial y}\right),$$

$$j_y = \frac{\sigma}{1+\beta^2}\left(\frac{\partial\psi'}{\partial x} - \beta\frac{\partial\phi'}{\partial x}\right).$$

Once the function F' is determined the currents, electric fields, and hence potentials at any point may be calculated. The function F' may be considered as a transform mapping the generator in the Z plane into the F' plane in which equipotentials of the field $\mathbf{E}'$ are straight lines of constant ϕ' and in which current stream lines are also straight such that $\psi' - \beta\phi'$ is constant along any one; i.e. they are at an angle $(\frac{1}{2}\pi - \theta)$ to the equipotentials. In order to determine F' the boundary conditions must first be examined.

On the insulating surfaces the normal component of current is zero;

$$j_y = 0, \tag{7}$$

or in terms of the field $\mathbf{E}'$

$$E'_y/E'_x = -\beta$$

on the electrodes

$$E'_x = -\partial\phi'/\partial x = 0. \tag{8}$$

Electrodes 1 and 3 are at the same electrostatic potential. By virtue of equation (4) this may be written

$$\phi'_1 = \phi'_3 + V_0 B_0 d. \tag{9}$$

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The current into electrode 3 equals that from electrode 1

$$-I_1 = I_3. \tag{10}$$

The total current into electrode 2 is the same as the total current from electrode 4 and this current is related to the potential difference between electrodes 2 and 4 through the load resistance.

$$-I_2 = I_4 = (\phi_4 - \phi_2)/R. \tag{11}$$

In terms of F' equation (11) can be written again by virtue of (4)

$$I_4 = \frac{\phi_4' - \phi_2' + V_0 B_0 d}{R}. \tag{12}$$

There are no currents flowing at $x = \pm\infty$. The problem is now specified completely and it merely remains to determine F'.

2. Formal solution of the problem

The four-electrode system is shown in figure 2 which also shows how currents may be expected to flow. The ends of electrode 1 are T_{11} and T_{12}, those of 2 T_{21} and T_{22}, etc. It is possible that there will be reversal of current on electrodes 2 and 4; the reversal points are marked A and B.

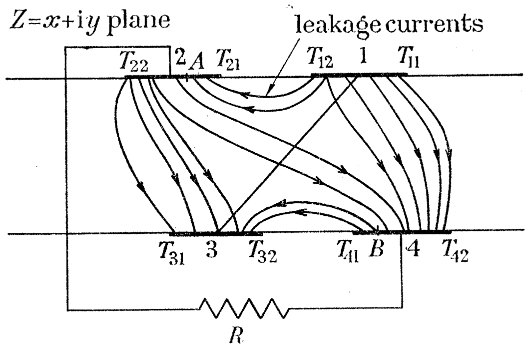

Figure 2.

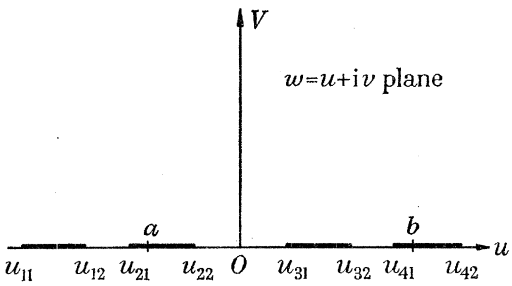

Figure 3.

A conformal mapping can be done in two stages. First the Z plane (figure 2) is mapped on to the $w(= u+\mathrm{i}v)$ plane (figure 3) by the transformation

$$w = \mathrm{e}^{\pi Z/d}. \tag{13}$$

The internal region of the duct in the plane is mapped on to the upper half of the w plane. Points u_{ij} correspond to the points T_{ij} in the Z plane and the current reversal points occur at $u = a, b$ respectively.

Next, the w plane is mapped into the F' plane by using a Schwarz–Christoffel transformation:

$$\frac{\mathrm{d}F'}{\mathrm{d}w} = K\,\mathrm{e}^{\mathrm{i}\theta}\,(w-a)\,(w-b)\prod_{n=1}^{4}(w-u_{n1})^{\theta/\pi-\frac{1}{2}}\,(w-u_{n2})^{-\theta/\pi-\frac{1}{2}}, \tag{14}$$

and since

$$\frac{\mathrm{d}w}{\mathrm{d}Z} = \frac{\pi}{d}\,w, \tag{15}$$

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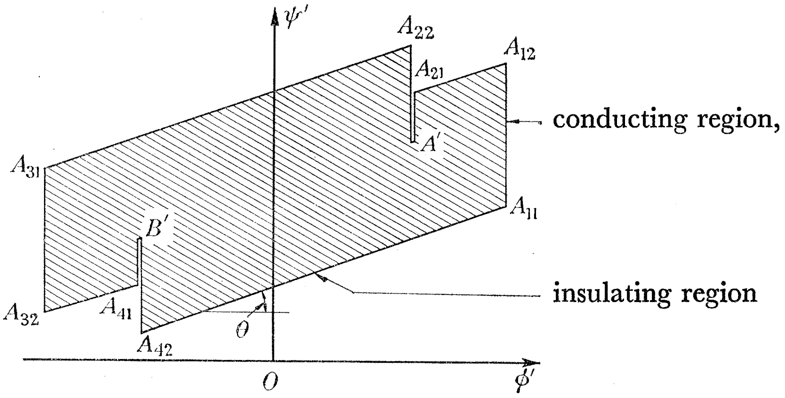
we have $$\frac{dF'}{dZ} = \frac{dF'}{dw}\frac{dw}{dZ} = \frac{K\pi}{d}\,e^{i\theta}\,w(w-a)(w-b)\prod_{n=1}^{4}(w-u_{n1})^{\theta/\pi-\frac{1}{2}}(w-u_{n2})^{-\theta/\pi-\frac{1}{2}}. \tag{16}$$

This transforms the interior of the duct in the Z plane into a decagon in the F' plane (figure 4). The boundaries of the duct correspond to the boundaries of the decagon; A_{ij} corresponds to T_{ij}, and A' and B' to A and B. The F' defined by this transformation satisfies all the boundary conditions completely, except that conditions (9), (10) and (12) have to be satisfied by the correct choice of K, a and b in equations (15) or (16), and if F' is to be determined completely, an arbitrary constant arising from integration of (16) is determined by using equation (4). The solution is unique and so the F' described here is the one sought.



$\theta = \arctan\beta \quad F' = \phi' + i\psi'$ plane

FIGURE 4.

Although the figures show current reversal on electrodes 2 and 4 this is not necessarily the case. The points a and b on the u axis of the w plane may appear on insulators which would mean a potential maximum would occur at corresponding points in the Z plane.

From equation (6) and the transform (16)

$$j = \frac{-\sigma}{(1+\beta^2)^{\frac{1}{2}}}\frac{\pi w}{d}K(w-a)(w-b)\prod_{n=1}^{4}(w-u_{n1})^{\theta/\pi-\frac{1}{2}}(w-u_{n2})^{-\theta/\pi-\frac{1}{2}}. \tag{17}$$

3. Determination of the constants

It is convenient to write

$$\frac{\sigma}{(1+\beta^2)^{\frac{1}{2}}}\frac{\pi}{d}K(w-a)(w-b) = R_1 + R_2 w + R_3 w^2, \tag{18}$$

and to determine the values of R_1, R_2 and R_3 from which K_1, a and b may be quickly found. Boundary condition (9) requires

$$0 = \phi_3' - \phi_1' + V_0 B_0 d.$$

Since ϕ' is constant along an electrode

$$\phi_3' - \phi_1' = \int_{T_{12}}^{T_{21}} d\phi' + \int_{T_{22}}^{T_{31}} d\phi'$$
$$= \mathscr{R}\left[\int_{u_{12}}^{u_{21}}\frac{dF'}{dw}\,du + \int_{u_{22}}^{u_{31}}\frac{dF'}{dw}\,du\right].$$

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Between u_{12} and u_{21} on the u axis (from 14)

$$\frac{\mathrm{d}F'}{\mathrm{d}w} = -\frac{d}{\pi\sigma}(1+\beta^2)^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}\theta}\,(R_1+R_2w+R_3w^2)\,\Psi,$$

where
$$\Psi = \prod_{n=1}^{4}|w-u_{n1}|^{\theta/\pi-\frac{1}{2}}\,|w-u_{n2}|^{-\theta/\pi-\frac{1}{2}},$$

which is always real. Therefore

$$\mathscr{R}\int_{u_{12}}^{u_{21}}\mathrm{d}\phi' = -\int_{u_{12}}^{u_{21}}\frac{d}{\pi\sigma}(R_1+R_2u+R_3u^2)\,\Psi\,\mathrm{d}u$$
$$= -(\mathrm{d}/\pi\sigma)\,(R_1G_{120}+R_2G_{121}+R_3G_{122}),$$

where
$$G_{ijk} = \int_{u_{i2}}^{u_{j1}} u^k\Psi\mathrm{d}u.$$

Similarly
$$\mathscr{R}\int_{u_{22}}^{u_{31}}\frac{\mathrm{d}F}{\mathrm{d}\omega}\mathrm{d}u = \frac{\mathrm{d}}{\pi\sigma}(R_1G_{230}+R_2G_{231}+R_3G_{232}),$$

and so condition (9) leads to

$$R_1(G_{120}-G_{230})+R_2(G_{121}-G_{231})+R_3(G_{122}-G_{232}) = VB\sigma\pi. \tag{19}$$

The current into electrode 1 is given by

$$\int_{T_{12}}^{T_{11}} j_y\,\mathrm{d}x = -\mathscr{I}\int_{T_{12}}^{T_{11}} j\,\mathrm{d}Z;$$

using equation (17) and reversing the direction of integration gives the current as

$$\mathscr{I}\int_{u_{11}}^{u_{12}}\frac{-\sigma}{(1+\beta^2)^{\frac{1}{2}}}K(w-a)\,(w-b)\prod_{n=1}^{4}(w-u_{n1})^{\theta/\pi-\frac{1}{2}}\,(w-u_{n2})^{-\theta/\pi-\frac{1}{2}}\,\mathrm{d}w,$$

and this becomes on further examination

$$= -\mathscr{I}\int_{u_{11}}^{u_{12}}\frac{d}{\pi}(R_1+R_2w+R_3w^2)\,\mathrm{e}^{\mathrm{i}(\theta-\frac{1}{2}\pi)}\,\Psi\,\mathrm{d}w$$
$$= \int_{u_{11}}^{u_{12}}\frac{d}{\pi}\frac{(R_1+R_2u+R_3u^2)}{(1+\beta^2)^{\frac{1}{2}}}\,\Psi\,\mathrm{d}u,$$
$$= \frac{d(R_1H_{10}+R_2H_{11}+R_3H_{12})}{\pi(1+\beta^2)^{\frac{1}{2}}}$$

when
$$H_{ij} = \int_{u_{i1}}^{u_{i2}} u^j\,\Psi\,\mathrm{d}u.$$

Doing a further integration over electrode 3 we find that condition (10) can be written

$$R_1(H_{10}+H_{30})+R_2(H_{11}+H_{31})+R_3(H_{12}+H_{32}) = 0. \tag{20}$$

By similar arguments condition (12) leads to

$$R_1\left[-R\frac{\sigma}{(1+\beta^2)^{\frac{1}{2}}}H_{20}+(G_{340}-G_{230})\right]+R_2\left[-R\frac{\sigma}{(1+\beta^2)^{\frac{1}{2}}}H_{21}+(G_{341}-G_{231})\right]$$
$$+R_3\left[-R\frac{\sigma}{(1+\beta^2)^{\frac{1}{2}}}H_{22}+(G_{342}-G_{232})\right] = VB\pi\sigma. \tag{21}$$

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The coefficients of R_1, R_2 and R_3 in equations (19), (20) and (21) can all be calculated using (13) to determine the electrode ends u_{ij} in the w plane. From the values of R_1, R_2, R_3 it is possible with the aid of (18) to find the constants K, a and b in the expression for $(\mathrm{d}F'/\mathrm{d}w)$ which allows calculation of current densities and integration of which gives electrostatic potential variation in the generator.

4. Conclusions

A solution has been obtained to the problem of determining the current pattern in a single-load generator with four electrodes. This solution takes into account the phenomenon of leakage current between electrodes on the same wall.

The results could be generalized so that a generator with any finite number n of electrodes could be considered. In this case there would be $2n$ Schwarz–Christoffel factors instead of eight as in equation (14) and there would be $n-1$ constants to determine instead of 3 as in this case.

Determination of the constants a and b makes it possible to calculate from equation (13) the actual coordinates of A and B in the z plane. Then it will be known which portion of the electrode is subjected to useful load current and which portion is allowing leakage current to flow. These calculations should therefore indicate the distance at which the electrodes should be set apart so that the leakage current is reduced to a suitable minimum, or to determine the conditions under which the leakage current vanishes.

If the leakage current can be neglected the coordinates a and b may be replaced by the coordinates of the electrode ends and the complicated determination of the constants may be omitted.

Reference (Nemkova & Alavidze)

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